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Note: These lecture notes are accompanied with video animations in a separate pdf file.

Conservative Systems

Consider $S\begin{cases}\dot{x} = f(x,y)\\ \dot{y} = g(x,y)\end{cases}$ - 2D dynamical system -

What is a trajectory?

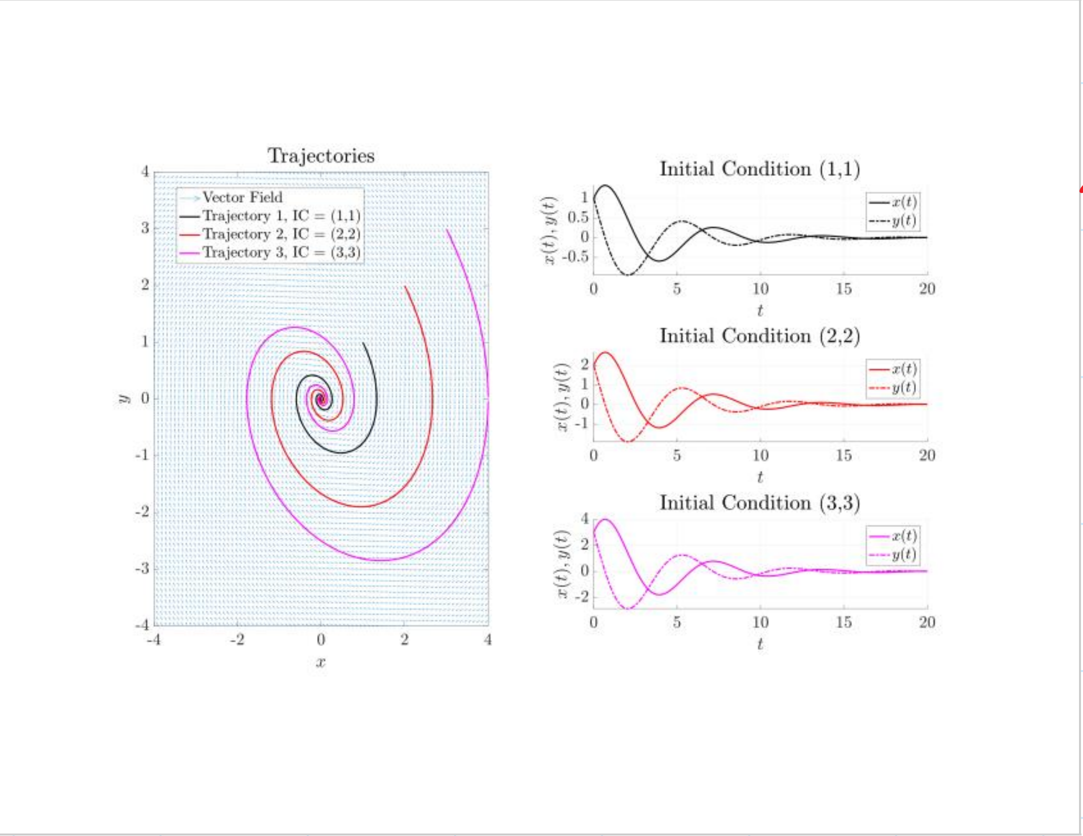
Think of a trajectory as the path of a ball with coordinates $(x(t), y(t))$ as time $t\uparrow$, where $x(t)$ and $y(t)$ are solutions to S.

So, for different initial conditions, you get different trajectories for the same dynamical system.

In 2D, it is easy to visualize trajectories using matlab.

e.g. 1: Consider $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & -1/2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$ eigenvalues : $\lambda = -\frac{1}{4} \mp i\frac{\sqrt{15}}{4}$

$\Rightarrow$ stable spiral.



simulations for 3 different initial conditions

Figure 1 shows an animation that illustrate the evolution of a trajectory in phase plane and in time

What is a closed Trajectory?

$\rightarrow$ Trajectory that "returns" to itself.

e.g. For linear 2D systems: Centers!

$\mathcal{M}$: $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ eigenvalues: $\mp i$

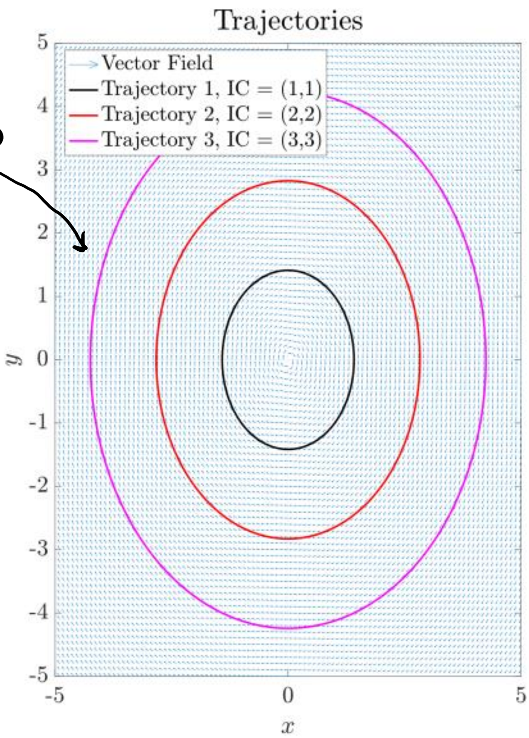
Define $E(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$

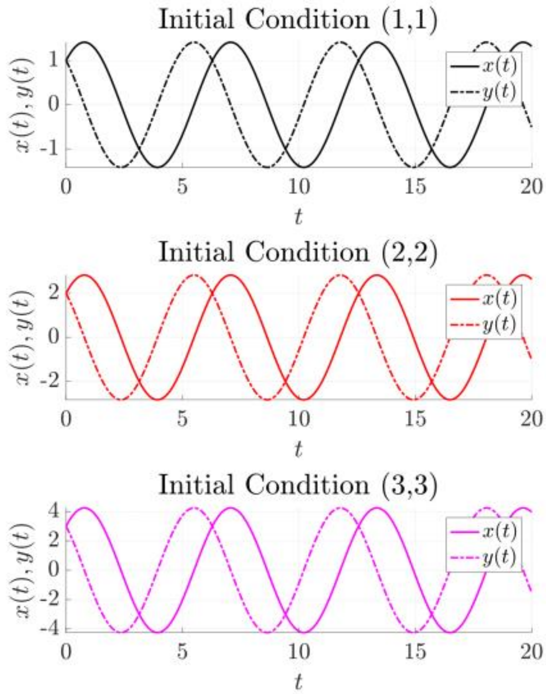
Evaluate E on a trajectory:

i.e. given $x(t)$ and $y(t)$ that are solutions, calculate:

$$E(x(t), y(t)) = \frac{1}{2}x^2(t) + \frac{1}{2}y^2(t) \quad \forall t.$$

closed trajectories





Observation: For each initial condition, $E(x(t), y(t))$ is constant in time.

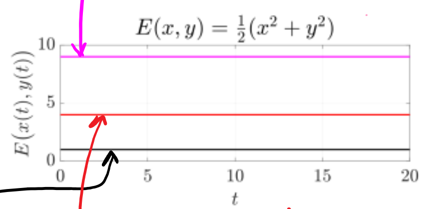
In fact:

$$E(x(t), y(t)) = \frac{1}{2}\left(x^2(t) + y^2(t)\right) = \frac{1}{2}\left(x^2(0) + y^2(0)\right)$$

$\therefore$ $E(x,y)$ is a conserved quantity

We call $\mathcal{M}$ a conservative system.

$x(0) = y(0) = 3$: $E(x(t), y(t)) = 9 \quad \forall t \geq 0$



$x(0) = y(0) = 2$: $E(x(t), y(t)) = 4 \quad \forall t \geq 0$

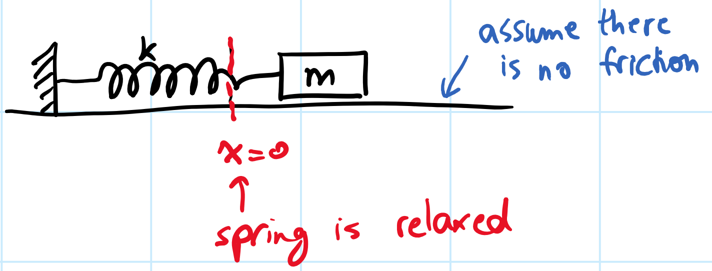
$x(0) = y(0) = 1$: $E(x(t), y(t)) = 1 \quad \forall t \geq 0$

So, $\mathcal{M}: \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ is a conservative system and

$E(x,y) := \frac{1}{2}(x^2 + y^2)$ is a conserved quantity.

But what does $E(x,y)$ mean physically?

physical meaning: The dynamical system $\mathcal{M}$ can be realized as a mechanical system composed of a box (of mass m) connected to a spring (of stiffness constant k).



Newton's Second Law: $F = ma$
Force
acceleration
mass

But $F = -kx$ and $a = \ddot{x}$

$\Rightarrow$ dynamics are given by: $m\ddot{x} = -kx$

for simplicity: let $m=1$ and $k=1$

$\Rightarrow$ $\ddot{x} = -x$ (second order linear differential equation)

Can be transformed to 2D linear dynamical system by introducing $y := \dot{x}$ (velocity)

$\Rightarrow$ $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ this is indeed system $\mathcal{M}$.

Total energy = kinetic energy + potential energy

$= \frac{1}{2} m \times \text{velocity}^2 + \frac{1}{2} k \times (\text{spring displacement})^2$

$= \frac{1}{2} m y^2 + \frac{1}{2} k x^2$

$= \frac{1}{2}(y^2 + x^2)$ $(m=1,\ k=1)$

$\hookrightarrow$ This is our $E(x,y)$!

$\therefore$ $E(x,y)$ is energy & it is conserved.

Consider $m\ddot{x} = F(x) \leftarrow$ some nonlinear force

potential energy $V(x)$: $F(x) = -\frac{dV(x)}{dx}$

substitute for $F(x)$: $m\ddot{x} = -\frac{dV(x)}{dx} \Rightarrow m\ddot{x} + \frac{dV(x)}{dx} = 0$

<u>useful trick:</u> multiply both sides by $\dot{x}$.

$$\Rightarrow m\dot{x}\ddot{x} + \frac{dV(x)}{dx}\dot{x} = 0$$

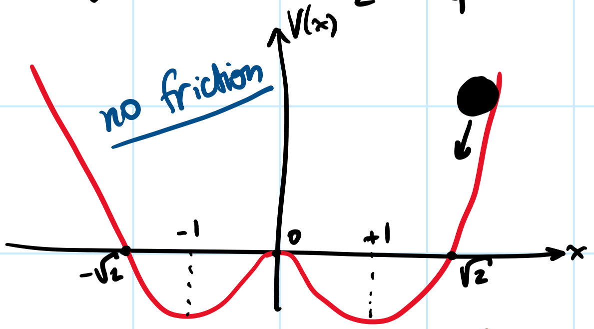
$$\boxed{\text{Aside: } \frac{d}{dt} f(x(t)) = \dot{x} f'(x(t))}$$

$$\Rightarrow \frac{d}{dt}\Big(\underbrace{\tfrac{1}{2}m\dot{x}^2 + V(x)}_{\text{Conserved quantity}}\Big) = 0$$

potential energy

kinetic energy

e.g. $V(x) = -\frac{x^2}{2} + \frac{x^4}{4}$



– Double-well potential –

$m\ddot{x} = F(x)$ $\quad F(x) := -\frac{dV(x)}{dx} = -x + x^3$

take $m = 1$

$\Rightarrow \ddot{x} = -x + x^3$

Conserved quanty is $E(x,y) = \frac{1}{2}\dot{x}^2 - \frac{x^2}{2} + \frac{x^4}{4}$

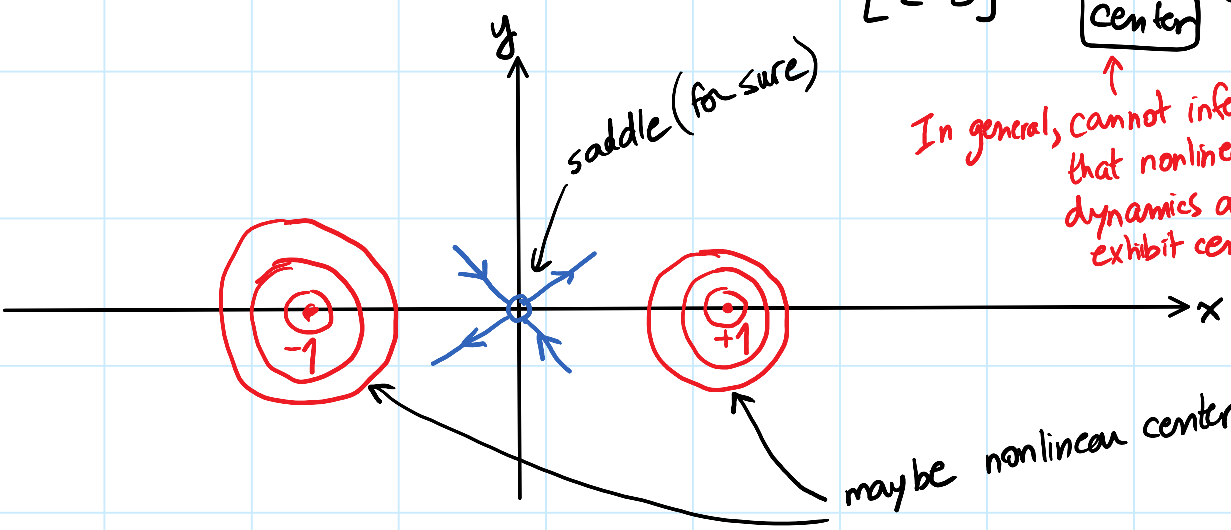
define $y := \dot{x} \Rightarrow \begin{cases} \dot{x} = y \\ \dot{y} = \ddot{x} = -x + x^3 \end{cases}$

Fixed points: $(x^*, y^*) \in \{(0,0), (0,+1), (0,-1)\}$

Linearize: $A = \begin{bmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{bmatrix}$ $\quad$ @ $(0,0)$ $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \lambda(A) = \mp 1$ $\boxed{\text{saddle}}$

@ $(0,\mp 1)$ $A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$ $\quad \lambda(A) = \mp i\frac{2}{\sqrt{2}}$ $\boxed{\text{center}}$

In general, cannot infer that nonlinear dynamics also exhibit centers.



Theorem (Nonlinear Centers for Conserved Systems)

Consider $\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$ where f and g are smooth functions.

Suppose that: (1) We know a conserved quantity $E(x,y)$

(2) (x^*, y^*) is an isolated fixed point.

(i.e. it is the only fixed point in its neighborhood)

If (x^*, y^*) is a local minimum of $E(x,y)$

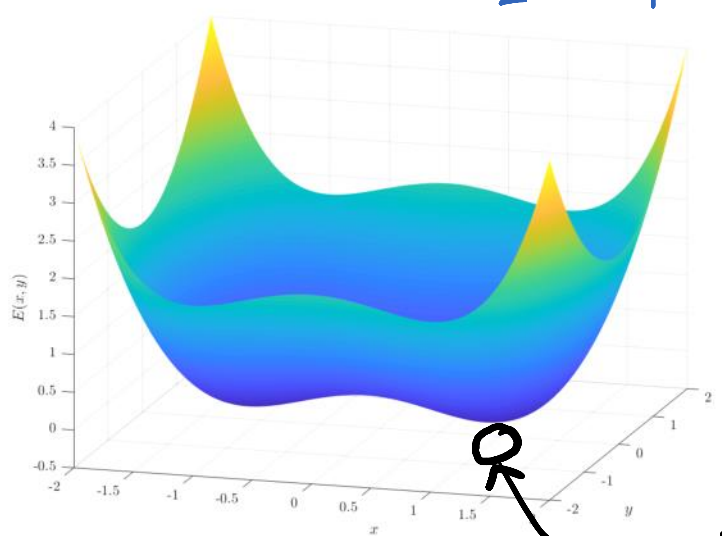
Then: All trajectories sufficiently close to the fixed point (x^*, y^*) are closed.

Let's apply the theorem on our double well example.

✓ (1) We know $E(x,y) = \frac{1}{2}y^2 - \frac{x^2}{2} + \frac{x^4}{4}$ is a conserved quantity

✓ (2) $(x^*, y^*) = (1,0)$ is an isolated fixed point in its neighborhood.

$$E(x,y) = \frac{1}{2}y^2 - \frac{x^2}{2} + \frac{x^4}{4}$$



✓ $(x^*, y^*) = (1,0)$ is a local minimum of $E(x,y)$

$\therefore$ Closed orbits exist around $(1,0)$

(Similar reasoning can be carried out for $(-1,0)$)

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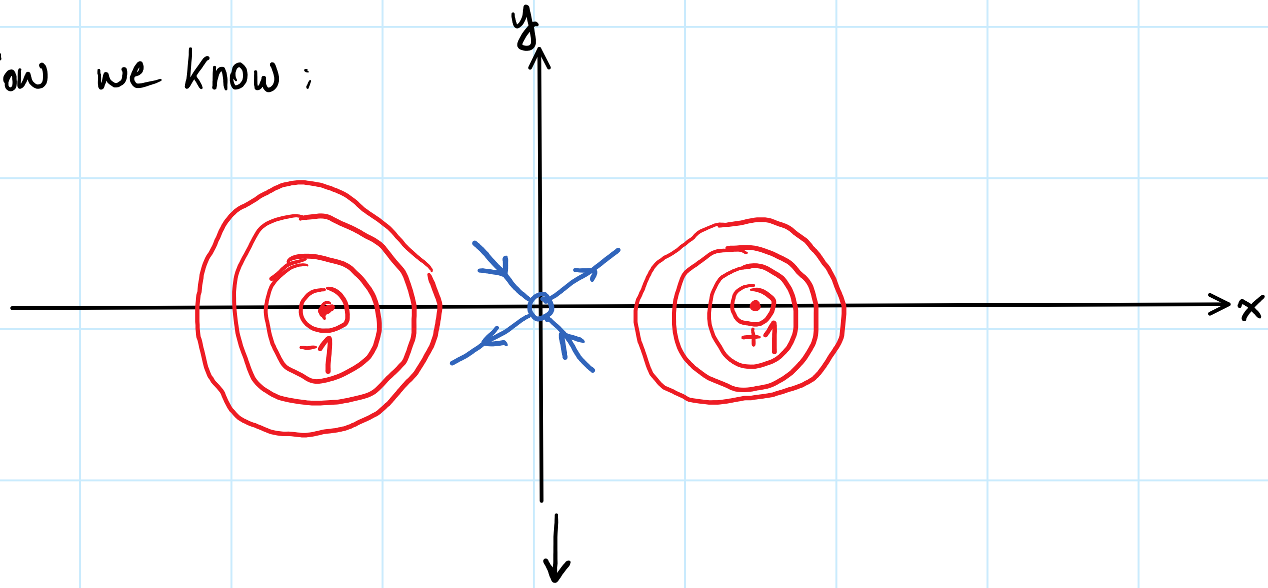
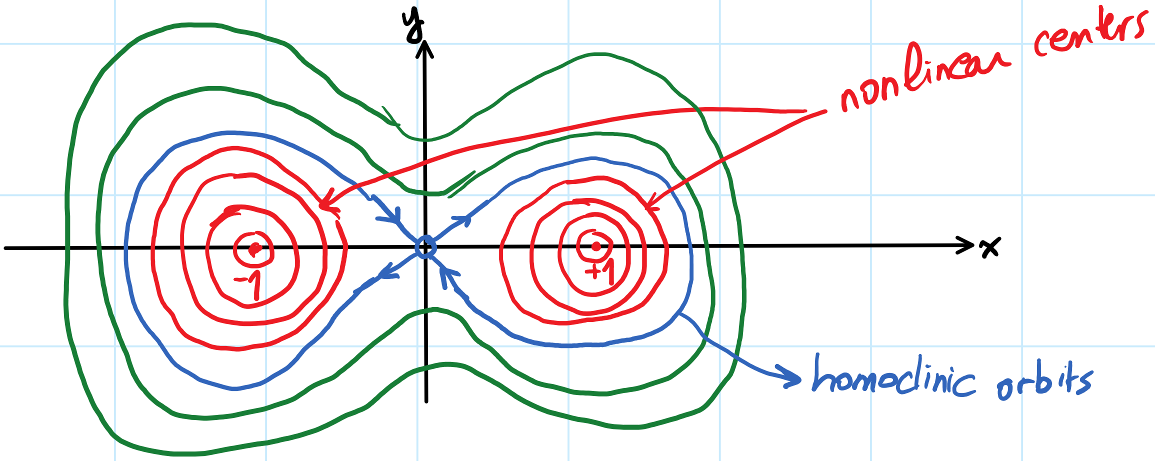
Now we know:





homoclinic orbits are trajectors that start and finish @ the same location in phase plane.

2 homoclinic orbits here:

See Figures 2, 3, 4 and 5 for animations for the double well example

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<u>Fact</u>: A conservative system <u>cannot</u> have an attracting fixed point.

<u>sketch of proof</u>:
- Assume (x^*, y^*) is fixed point (attractive)
- Assume that E is conserved